The averaged null energy condition and difference inequalities in quantum field theory

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Abstract

For a large class of quantum states, all local (pointwise) energy conditions widely used in relativity are violated by the renormalized stress-energy tensor of a quantum field. In contrast, certain nonlocal positivity constraints on the quantum stress-energy tensor might hold quite generally, and this possibility has received considerable attention in recent years. In particular, it is now known that the averaged null energy condition-the condition that the null - null component of the stress-energy tensor integrated along a complete null geodesic is nonnegative for all states- holds quite generally in a wide class of spacetimes for a minimally coupled scalar field. Apart from the specific class of spacetimes considered (mainly two-dimensional spacetimes and four-dimensional Minkowski space), the most significant restriction on this result is that the null geodesic over which the

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average is taken must be achronal. Recently, Larry Ford and Tom Roman have explored this restriction in two-dimensional flat spacetime, and discovered that in a fiat cylindrical space, although the stress-ellcrgy tensor itself fails to satisfy the averaged null energy condition (ANEC) along the (non-achronal) null geodesics, when the "Casimir-vacuum" contribution is subtracted from the stress- c]lcrg,y the resulting tensor dots satisfy the ANEC inequality. Ford and Roman name this class of constraints on the quantum stress-c]lcrgy tensor "difference inequalities." Here 1 give a proof of the difference inequality for a minimally coupled massless scalar field in an arbitrary two-d imensional spacetime, using the same techniques as those wc relied on to prove ANEC in an earlier paper with Robert Wald. 1 begin with an overview of averaged energy conditions in quantum field theory.

1. Averaged energy conditions

All general results in relativity require some information about the matter content of spacetime as input. Although in classical physics the issue can be avoided by assuming spacetime to be empty, in quantum theory absolute vacuum is meaningless, and questions about how matter fields contribute to the semiclassical Einstein equations are unavoidable whenever quantum effects are important gravitationally. In classical relativity, information about the matter content of spacetime can often be specified quite succinctly by the form of the stress-energy tensor for a specific matter field. For example, the electromagnetic field contribution,

$$T_{ab} = \frac{1}{4\pi}\left(F_a{}^c F_{bc} - \frac{1}{4} g_{ab} F^{cd} F_{cd}\right), \tag{1}$$

can be characterized by stating that the stress-energy tensor has the form Eq. (1) for some closed two-form F_{ab} that satisfies the vacuum Maxwell's equation $d*F=0$. Similar characterizations exist in principle in quantum field theory. For example, the regularized stress-energy tensor of a minimally-coupled scalar field on Minkowski spacetime can be characterized as any tensor $\langle T_{ab}\rangle$ that has the form of the coincidence limit

$$\langle T_{ab}\rangle(x) = \lim_{y\to x}\left[\frac{\partial^2}{\partial x^a \partial y^b} w(x,y) - \frac{1}{2} g_{ab}\left(g^{cd}\frac{\partial^2}{\partial x^c \partial y^d} + m^2\right) w(x,y)\right] \tag{2}$$

in an inertial coordinate system $\{x^a\}$, where $w(x,y)$ is any smooth, symmetric bi-solution of the wave equation such that the two-point function $\mu_0 + w$ - where μ_0 is the symmetric two-point function of the Poincare vacuum state satisfies the positivity condition (see Sect. 3 below). Already in flat spacetime, a description of the quantum matter content based on Eq. (2) is far more complicated than the corresponding classical description as illustrated by Eq. (1). In arbitrary curved spacetime, the kind of characterization of $\langle T_{ab}\rangle$ expressed in Eq. (2), although available in principle, is so heavily cluttered with formalism as to be essentially intractable.

In fact, even in classical physics a complete, detailed specification of the stress-energy as in Eq. (1) is seldom very illuminating unless one's primary interest is to find exact solutions of Einstein's equations, Instead, in modern approaches to relativity only the most fundamental features of T_{ab}, distilled from expressions like Eq. (1), are relied on to gain general insight into spacetime structure. These fundamental features are that T_{ab} is a conserved, symmetric tensor, and that T_{ab} satisfies certain "energy conditions" at every point in spacetime; the energy conditions express, roughly, the idea that the locally-measured energy density must be positive everywhere for all observers. The energy conditions (or, more precisely, at least the weak energy condition) are universal in the sense that (i) they are obeyed by the classical stress-energy tensors of all matter fields, and (ii) they play a crucial role in deriving most of what we know about the large-scale structure of spacetime. Indeed, it is perfectly plausible to regard the energy conditions, along with the conservation property, as a complete characterization of classical matter in genera] relativity,

In deep contrast with this classical picture, in quantum theory no such compact characterization is yet available for the right hand side of the semi-classical Einstein equations. The symmetry and conservation prop erties still hold for $\langle T_{ab}\rangle$, of course, but none of the local energy conditions do. Even in flat, Minkowski space, the regularized (norlnal-ordered) expectation value $\langle\omega|{:}T_{00}(x){:}|\omega\rangle$ at any point x is unbounded from below as a functional of the quantum state ω. One might think that this is a pathology, as is often the case in quantum field theory, that stems from localizing the operator $:T_{00}(x):$ to a single point x in spacetime. It turns out, however, that the volume integral of $\langle\omega|{:}T_{00}(x){:}|\omega\rangle$ over any fixed, spacelike 3-box of *finite* size is also unbounded from below as a functional of ω (and a similar result holds for the spacetime-volume integral over a compact 4-box). It appears that by choosing the quantum state ω appropriately one can stuff an unbounded amount of negative energy into any fixed, finite region of spacetime, possibly at the expense of placing more and more positive energy outside the sharply defined boundaries of that region ([1]).

In the absence of a workable, complete characterization of regularized stress-energy tensors in quantum field theory, many of the basic questions about global spacetime structure in semiclassical gravity remain unanswered. For example, can spacetime singularities, generically unavoidable with classical matter, be avoided when quantum effects make the dominant contribution to stress-energy? Are classically forbidden configurations of spacetime curvature (such as traversable wormholes, certain kinds of topology change) allowed in semiclassical gravity? Is the total mass of a bounded lump of quantized matter positive as measured from infinity? More generally, is any conserved tensor T_{ab} realizable as the regularized stress-cllcrgy tensor of some quantum state? If this were the case, semiclassical gravity would have almost no physical content. If it is not the case, then what are the constraints T_{ab} has to satisfy to be a physical stress- energy tensor? I would like to advocate the discovery of a useful and complete characterization of quantum stress-encn-gy tensors as one of the most important unsolved problems in curved spacetime quantum field theory.

Promising first steps towards the construction of such a characterization have appeared in recent years, beginning with [2], aud with further developments in [3]- [8]. Au earlier work, Ref. [9], discussed somewhat related issues. The results of these early investigations concern various nonlocal constraints on the stress-cncrg,y tensor involving its integrals along causal geodesics; the most important of these constraints is the averaged null energy condition (A NEC). In its simplest form, ANEC constrains the stress-energy tensor T_{ab} such that the integral

$$\int_\gamma T_{ab}\, k^a\, k^b\, dv \geqslant 0 , \tag{3}$$

where γ is a complete *null* geodesic with affine parameter v and corresponding tangent vector k^a. The precise, general formulation of ANEC which dots not assume the convergence of the integral in Eq. (3) cau be found in Sect.2 of [4].

For a number of significant global results in relativity, ANEC (or at least a corresponding condition along half-complete null geodesics) seems to be strong enough to replace the classical pointwise null energy condition; these include the Penrose singularity theorem ([10]-[12]), and the positive mass theorem ([13]). For a minimally coupled scalar field in two dimensions, it is known that the regularized stress-energy tensor satisfies ANEC with complete generality, along all complete achronal null geodesics in any globally-hyperbolic spacetime, and in every Hadamard quantum state of the field ([4]). In four dimensions, this general ANEC result holds in Minkowski spacetime, and, more generally, in any spacetime with a bifurcate Killing horizon it holds along the achronal null generators of the horizon, provided an isometry-invariant (with respect to the Killing field) Hadamard state exists (see [4] for details).

Conditions similar to ANEC but with γ replaced with a complete *time-like* geodesic hold with some generality for quantum stress-energy tensors in Minkowski space ([2], [6], [7]). However, this appears to be a special feature of flat-spacetime quantum field theory; it is not difficult to find curved-spacetime counterexamples to the timelike averaged weak energy condition ([14]). In fact, relying on a simple scaling argument, we pointed out in [4] that even ANEC, although it holds with complete generality in two dimensions, cannot hold generally in curved four-dimensional spacetimes (see [16] for a further development of this scaling idea). Although this argument and its conclusion are correct, it is by no means clear that they spell the demise of averaged energy conditions in quantum field theory; that many people have been led to believe so appears to be the result of a greatly exaggerated interpretation of the argument. Elsewhere ([17]) I argue for a more moderate interpretation, based, essentially, on a generalized version of ANEC in which the right hand side of Eq. (3) is replaced with a more general finite lower bound. I do regard ANEC-type inequalities as a very promising starting point towards the formulation of a complete set of universal constraints that characterize quantum stress-energy tensors. For example, one might think that ANEC is too weak because it only constrains the integrals of T_{ab} along null geodesics; in fact, ANEC appears to be powerful enough to place constraints on other, more general averages of the stress-energy tensor. I will now present a simple analysis that supports this view:

I will consider a stress-energy tensor T_{ab} that satisfies the simple form of ANEC given by Eq. (3); therefore I implicitly assume that T_{ab} falls off appropriately at infinity. A more sophisticated analysis that does away with this assumption (and possibly with some of the other strong assumptions I will make below) can probably be given; I retain these assumptions to keep my analysis transparent. More precisely, I consider a globally hyperbolic spacetime $(\mathcal{M}, g)$, and a conserved stress-energy tensor T_{ab} which satisfies ANEC in the form of Eq. (3) along all complete, achronal null geodesics in $(\mathcal{M}, g)$. Let $\Sigma \subset \mathcal{M}$ be a spacelike Cauchy surface, and $S \subset \Sigma$ be a compact submanifold (with boundary) in Σ. I assume:

(A1) The subregion $S \subset \Sigma$ is chosen large enough such that ANEC [Eq. (3)] holds along the null generators of the future horizon $H^+(S)$.

Note that the generators of H^{+} (S) arc necessarily achronal null geodesics; they have their past endpoints 011 the boundary ∂S of S, and their future endpoints on a (in general complicated) caustic set $\mathfrak{C}$. 1 assume that:

(A2) There exists a time function α defined on the domain of dependence $D^{+}(S)$ such that (i) $\alpha = O$ *on* S and $\alpha = 1$ 011 $H^{+}(S)$, (ii) $\alpha_{;a} = - \kappa n_a$ 011 S, where n^a is the future-pointillg unit normal to S and κ is a positive constant, and (iii) throughout the interior of D^{+} (S)

$$T^{ab}\alpha_{;ab} \leqslant q\, T^{ab}\alpha_{;a}\alpha_{;b} \tag{4}$$

for some constant $q > 0$.

Theorem: Under the assumptions (A1) and (A2), the total energy contained in the region $S \subset \Sigma$ is nonnegative:

$$\int_S T^{ab} n_a n_b \, d^3\sigma \geqslant 0, \tag{5}$$

where $d^3\sigma$ is the volume element 011 Σ.

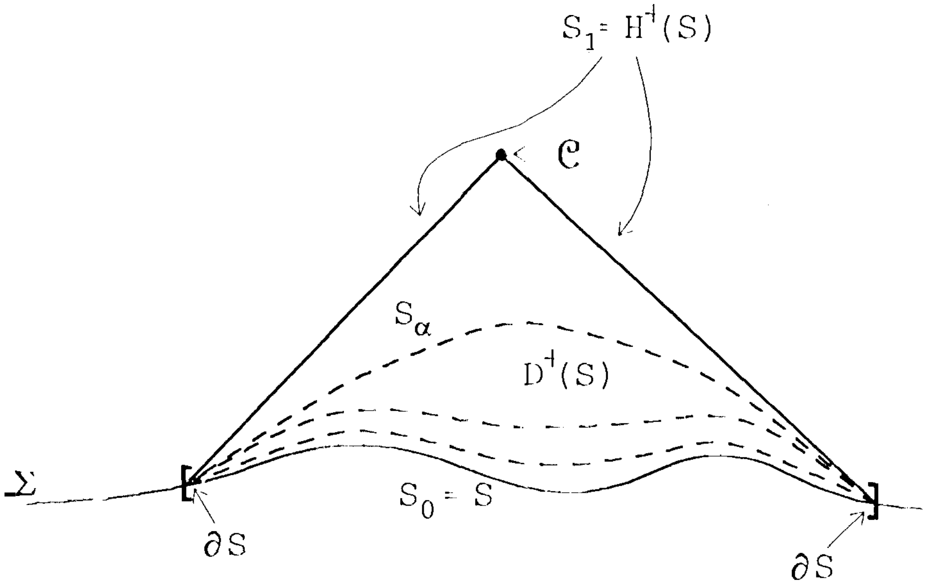

Figure 1. The 'geometry involved in the analysis leading to **Eq.** (5). **The spacelike** surfaces S_α are the level **sets** of the time function α; accordingly, SO is S and S_1 is the future horizon H^{+} (S).

The assumption (A 1) is quite reasonable within the scope of the present discussion); only the last part of assumption (A2) [i.e., the inequality Eq. (4)] is unpleasantly strong. (See Fig. 1 for a two-dimensional representation of the geometry involved in this analysis). Notice that T_{ab} dots not necessarily satisfy the (pointwise) weak energy condition, so the ri.gilt hand side of Eq. (4) is not necessarily positive, and simply choosing a large $q > 0$ will not do. Moreover, because of its construction the time function α has to develop gradient singularities at the boundary ∂S and at the caustic set $\mathcal{C}$ [but α is smooth throughout the (open) interior of the domain of dependence $D^+(S)$], so Eq. (4) effectively constrains the asymptotic behavior of T_{ab} near $\mathcal{C}$ and ∂S (see Fig. 1). For a typical example of the geometry involved here consider two-dimensional Minkowski spacetime, with the surface Σ given by $\{t = -K\}$, $K > 0$, and with $S \subset \Sigma$ given by that piece of Σ lying within the wedge $\{|x| \leqslant |t|\}$. In this example the horizon $H^+(S)$ is the past null cone of the origin, $H^+(S) = \{|t| = |x|, -K \leqslant t \leqslant 0\}$, and for $|x| << |t|$ the time function α can be taken as

$$\alpha = \frac{1}{K}\left(K + t - \frac{x^2}{t}\right)$$

up to corrections of order (x^2/t^2). This function satisfies the conditions (i) and (ii) of (A2) [again up to order (x^2/t^2)] with $\kappa = 1/K$. Then, assuming that T_{00} is the dominant component (in absolute value) of T_{ab},

$$T^{ab}\alpha_{;ab} \approx T^{00}\alpha_{,tt} = -T^{00}\frac{2}{Kt}\frac{x^2}{t^2},$$

$$T^{ab}\alpha_{;a}\alpha_{;b} \approx T^{00}\alpha_{,t}^2 = T^{00}\frac{1}{K^2}\left(1 + \frac{x^2}{t^2}\right)$$

Therefore, in the region $|x| \ll |t|$ throughout the domain of dependence $D^+(S)$, the quantity $T^{ab}\alpha_{;ab}$ has the same sign as $T^{ab}\alpha_{;a}\alpha_{;b}$ and is down in magnitude by a factor x^2/t^2. With the provision that T^{ab} falls off nicely at the boundary ∂S and near the caustic set (here the origin) $\mathcal{C}$ (where the true α becomes singular), and with the constant q chosen in the range $0 < q < 1$, this Minkowski-spacetime example suggests the inequality Eq. (4) to be a reasonable assumption. Quite possibly this inequality can be weakened considerably without changing the main argument in the proof of the theorem, which I now proceed to give.

Proof of the **theorem**: Let me define (see Fig. 1)

$$S_u \equiv \{p \in D^+(S) \mid \alpha(p) = u\},$$
$$V_u \equiv \{p \in D^+(S) \mid \alpha(p) \leqslant u\}.$$

By integrating the identity

$$(T^{ab}\alpha_{;a})_{;b} = T^{ab}\alpha_{;ab}$$

over the volume V_u, I obtain

$$-\int_{S_u} T^{ab}\alpha_{;a}n_b\, d^3\sigma + \int_{S_0} T^{ab}\alpha_{;a}n_b\, d^3\sigma = \int_{V_u} T^{ab}\alpha_{;ab}\, dV, \qquad (6)$$

where n_b denotes the future-pointing unit normal and $d^3\sigma$ is the volume form on S_u. Putting

$$I(u) \equiv -\int_{S_u} T^{ab}\alpha_{;a}n_b\, d^3\sigma, \qquad (7)$$

and combining Eq. (6) with Eq. (4), I obtain the inequality

$$I(u) \leqslant I(0) + q\int_{V_u} T^{ab}\alpha_{;a}\alpha_{;b}\, dV. \qquad (8)$$

Since $\alpha_{;b}$ is parallel and opposite to the future-pointing unit normal n_b, the second term on the right hand side of Eq. (8) can be written as

$$\int_{V_u} T^{ab}\alpha_{;a}\alpha_{;b}\, dV = - \int_0^u d\alpha \int_{S_\alpha} T^{ab}\alpha_{;a} n_b\, d^3\sigma$$
$$= \int_0^u d\alpha\, I(\alpha)\ ; \qquad (9)$$

hence the inequality Eq. (8) takes the form

$$I(u) \leqslant I(0) + q \int_0^u I(\alpha)\, d\alpha\ . \qquad (10)$$

Now inspect the definition Eq. (7) of the quantity $I(u)$. On the null surface $\{\alpha = 1\} = H^+(S)$, $\alpha^{;a}$ (since it is a null gradient) is necessarily an affine tangent vector along the null geodesic generators. Therefore, $I(1)$ is the average over the set of null generators of $H^+(S)$ of the ANEC integrals of T_{ab} along those generators. Thus $I(1) \geqslant 0$ by assumption (A1). But it is clear from the inequality Eq. (10) that the conclusion $I(1) \geqslant 0$ is incompatible with the assumption $I(0) < 0$. Therefore I conclude that $I(0) \geqslant 0$. By assumption (A2)-(i) and (A2)-(ii)

$$I(0) = \kappa \int_S T^{ab} n_a n_b\, d^3\sigma\ ,$$

and the assertion of the theorem [Eq. (5)] follows. □

2. Difference inequalities

As I mentioned briefly in the previous section, ANEC needs to be generalized since in the strict form given by Eq. (3) it is typically violated in four-dimensional curved spacetimes ([4], [16]). The most natural generalization of ANEC involves replacing the right hand side of Eq. (3) with a broadly specified lower bound; I will discuss this in more detail in [17]. Another, closely related generalization has been discovered by L. Ford and T. Roman in [18]; it involves what they term "difference inequalities." A difference inequality is an ANEC-type inequality of the form

$$\int_\gamma \left(\langle\omega|T_{ab}|\omega\rangle - D_{ab} \right) k^a k^b\, dv \geqslant 0 \quad \forall\omega\ , \qquad (11)$$

where $\langle\omega|T_{ab}|\omega\rangle$ denotes the renormalized stress-energy tensor in the quantum state ω, D_{ab} is a state-independent, geometric tensor on spacetime, and the integral is evaluated along a complete null geodesic γ as in Eq. (3). The difference inequality can be given a more precise formulation that does not require the convergence of the integral in Eq. (11) in just the same way as ANEC [see Sect. 2 of [4] and Eq. (15) below]. If the integral

$$\int_\gamma D_{ab}\, k^a k^b\, dv \qquad (12)$$

converges, Eq. (11) yields a (in general nonzero) lower bound on the ANEC integral:

$$\int_\gamma \langle\omega|T_{ab}|\omega\rangle\ k^a k^b\ dv \geqslant \int_\gamma D_{ab}\ k^a k^b\ dv \qquad \forall\omega\ . \qquad (13)$$

It is this form of the difference inequality which makes it potentially significant for applications such as singularity theorems (see [17]).

What Ford and Roman have discovered in []8] is that for a massless Klein-Gordon field on the flat cylinder (two-dimensional Minkowski spacetime identified modulo a discrete group of spatial translations), the difference inequality Eq. (1 1) holds along all complete null geodesics provided D_{ab} is the stress-energy tensor of the Casimir vacuum state:

$$D = \frac{\pi}{6L^2}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{14}$$

where L is the length of the spatial sections (i. e., the points (x,t) and $(x+L,t)$ in Minkowski spacetime arc identified). Note that complete null geodesics on the cylinder arc *not* achronal, and ANEC is violated along them [e.g. in the Casimir vacuum state where $\langle T_{ab}\rangle$ is given by the right hand side of Eq. (14)], Also note that the difference inequality cannot have the form Eq. (13) in this case since the integral Eq. (1'2) dots not converge. Nevertheless, this two-dimensional difference-inequality result of [18] relaxes the achronality assumption of the ANEC theorem proved in [4], and its main significance lies in this improvement.

In the remaining sections of the paper 1 will give a general proof of the difference inequality for a massless Klein-Gordon field in an arbitrary, globally hyperbolic two-dimensional curved spacetime $(\mathcal{M}, g)$. More precisely, for every Hadamard state ω of the field and for every complete null geodesic γ in $(\mathcal{M}, g)$ with affine parameter $v \in (-\infty, \infty)$, 1 will prove that the following holds: Let $c(x)$ be any bounded real-valued function of compact support on $\mathbb{R}$ whose Fourier transform $\hat{c}(k)$ is such that for some $\delta > 0$ the function $(1+k^2)^{1+\delta}|\hat{c}(k)|$ is bounded [i.e., $|\hat{c}(k)|$ decays at least as fast as $|k|^{-2-2\delta}$ as $|k| \to \infty$; this implies that $c(x)$ is C^1.] Then, for all choices of origin of the affine parameter v, the regularized stress-energy tensor $\langle\omega|T_{ab}|\omega\rangle$ satisfies the inequality

$$\liminf_{\lambda\to\infty} \int_{-\infty}^{\infty} \left(\langle\omega|T_{ab}|\omega\rangle - D_{ab} \right) k^a k^b \,[c(v/\lambda)]^2\, dv \geqslant 0\,, \tag{15}$$

where D_{ab} is a state-independent tensor which 1 will specify [D_{ab} depends only on the geometry of $(\mathcal{M}, g)$]. When the integrand $(\langle\omega|T_{ab}|\omega\rangle - D_{ab})\, k^a k^b$ is integrable, Eq. (15) reduces to the simple form Eq. (1 1) of the difference inequality.

My proof of the difference inequalit y Eq.(15) will be entirely parallel to the proof of ANEC in two dimensions that wc gave in Sects. 4 and 5 of [4], and I will omit all details which arc simply restatements of the corresponding details in [4] modified to suit the present analysis. Consequently, readers who wish to follow the remaining sections of this paper closely will find it useful to have a copy of [4] at hand while they do so. 1 will begin, in Sect. 3 below, by discussing the relationship between quantum states on Minkowski spacetime $\mathbb{R}^2$ and quantum states on the flat cylinder $S^1 \times \mathbb{R}$. 1 describe the proof of the difference inequality for the flat cylinder $S^1 \times \mathbb{R}$ in Sect. 4, and for a general, curved two-dimensional spacetime in Sect. 5.

3. States on $\mathbb{R}^2$ and states on $S^1 \times \mathbb{R}$

As we did throughout [4], so also here I will adopt the algebraic viewpoint on quantum field theory in a curved (globally hyperbolic) spacetime $(\mathcal{M}, g)$. In particular, quantum states ω are specified by their two-point distributions

$$\lambda[F, G] \equiv \omega(\phi[F]\phi[G]) , \qquad F, G \in S(\mathcal{M}) , \tag{16}$$

$$\lambda(f, g) \equiv \lambda[Ef, Eg] , \qquad f, g \in C_0^\infty(\mathcal{M}) , \tag{17}$$

where $S(\mathcal{M})$ is the space of all solutions of the Klein-Gordon equation which are compact supported on Cauchy surfaces, and $Ef \in S(\mathcal{M})$ demotes the "advanced minus retarded" solution with source $f \in C_0^\infty(\mathcal{M})$ (see Sect. 3 of [4] for a concise introduction to the algebraic approach). The two-point function λ can be written in the form

$$\lambda = \mu + \tfrac{1}{2} i \sigma , \tag{18}$$

where $\mu[F, G] = \mathrm{Re}(\lambda[F, G])$, and $\sigma[F, G]$ is the Klein-Gordon inner product (see Sect. 3 in [4] for details). The algebraic positivity condition on the state ω is equivalent to the inequalities $\mu[F, F] \geqslant 0 \; \forall F \in S(\mathcal{M})$, and

$$\mu[F, F]\, \mu[G, G] \geqslant \tfrac{1}{4} |\sigma[F, G]|^2 \qquad \forall F, G \in S(\mathcal{M}) \tag{19}$$

Note that Eq. (19) implies $\mu[F, F] > 0$ for all $F \neq 0$ in $S(\mathcal{M})$.

Let $\mathbb{M}$ denote the two-dimensional Minkowski spacetime $(\mathbb{R}^2, \eta)$, where $\eta = dx^2 - dt^2 = - du\, dv$, and let $\mathbb{L}$ denote the flat cylinder $S^1 \times \mathbb{R}$ obtained from Minkowski space by the identification $(x, t) \equiv (x + L, t)$. There exists a canonical "wrapping" map $\mathcal{W} : C_0^\infty(\mathbb{M}) \to C_0^\infty(\mathbb{L})$ given by

$$\mathcal{W} : f(x, t) \mapsto \sum_{n=-\infty}^{\infty} f(x + nL, t) , \qquad f \in C_0^\infty(\mathbb{M}) , \tag{20}$$

and a corresponding "wrapping" map $\mathcal{W}_S : S(\mathbb{M}) \to S(\mathbb{L})$ which satisfies $\mathcal{W}_S \circ E_M = E_L \circ \mathcal{W}$, where E_M and E_L denote the advanced-minus-retarded Green's functions on $\mathbb{M}$ and $\mathbb{L}$, respectively. I will denote $\mathcal{W}_S$ by the same symbol as $\mathcal{W}$ as long as it is clear from the context which map is which. Also, with $f \in C_0^\infty(\mathbb{M})$ and $F \in S(\mathbb{M})$, I will use the shorthand notation f^W and F^W to demote $\mathcal{W}(f) \in C_0^\infty(\mathbb{L})$ and $\mathcal{W}(F) \in S(\mathbb{L})$, respectively. Note that the map $\mathcal{W}$ is onto both from $C_0^\infty(\mathbb{M})$ to $C_0^\infty(\mathbb{L})$ and from $S(\mathbb{M})$ to $S(\mathbb{L})$.

Let ϕ be a massless Klein-Gordon field on $\mathbb{M}$, and similarly on $\mathbb{L}$. A Hadamard ([4]) quantum state ω on $\mathbb{M}$ is specified by a two-point function of the form

$$\mu_M(x, x') = \mu_{0\,M}(x, x') + w_M(x, x') , \tag{21}$$

where μ_{0M} is the two-point function of the Poincare vacuum [given by Eq. (19) in [4], Sect. 4], and w_M is a smooth, symmetric bi-solution of the Klein-Gordon equation such that $\mu_M = \mu_{0M} + w_M$ satisfies the positivity inequality

$$\mu_M(f, f)\, \mu_M(g, g) \geqslant \tfrac{1}{4} |\sigma_M(f, g)|^2 \qquad \forall f, g \in C_0^\infty(\mathbb{M}) . \tag{22}$$

On the flat cylinder $\mathbb{L}$, the analogue of the Poincare vacuum state is the Casimir vacuum, which can be constructed, e.g., by a mode decomposition where positive-frequency solutions are defined with respect to the canonical timelike Killing vector on $\mathbb{L}$. The Casimir vacuum is a Hadamard state and I will denote its two-point function by μ_{0L} [see Eq. (28) below]. The two-point function of any other Hadamard state on $\mathbb{L}$ can be written in the form

$$\mu_L(x,x') = \mu_{0L}(x,x') + w_L(x,x') , \tag{23}$$

where w_L is a smooth, symmetric bi-solution of the Klein-Gordon equation on $\mathbb{L}$ such that μ_L satisfies the positivity inequality appropriate for $\mathbb{L}$:

$$\mu_L(f,f)\,\mu_L(g,g) \geqslant \tfrac{1}{4}|\sigma_L(f,g)|^2 \qquad \forall f,\ g \in C_0^\infty(\mathbb{L}) . \tag{24}$$

What is the relationship between Hadamard states on $\mathbb{L}$ and Hadamard states on the Minkowski spacetime $\mathbb{M}$? To explore this question, it is convenient to pull back distributions defined on $\mathbb{L}$, (such as μ_L and al,) via the wrapping map $\mathcal{W}$ so that they become distributions on $\mathbb{M}$. For example, the pull-back of the distribution μ_L is the distribution ${\mu_L}^M$ on $\mathbb{M}$ defined by

$$ {\mu_L}^M(f,g) \equiv \mu_L(f^W,g^W) , \qquad f,\ g \in C_0^\infty(\mathbb{M}) , \tag{25}$$

and the pull-backs ${\mu_{0L}}^M$, ${w_L}^M$ and ${\sigma_L}^M$ of the distributions μ_{0L}, w_L and σ_L are defined similarly. Since the wrapping map $\mathcal{W}$ is onto, I can now rewrite the positivity condition for w_L [Eq. (24)] as an inequality defined purely on $\mathbb{M}$:

$$({\mu_{0L}}^M + {w_L}^M)(f,f)\,({\mu_{0L}}^M + {w_L}^M)(g,g) \geqslant \tfrac{1}{4}|{\sigma_L}^M(f,g)|^2 \qquad \forall\ f,\ g \in C_0^\infty(\mathbb{M}) . \tag{26}$$

Is ${\mu_L}^M = {\mu_{0L}}^M + {w_L}^M$ the two-point function of a Hadamard state on Minkowski spacetime? Clearly, ${\mu_L}^M$ in general, and ${\mu_{0L}}^M$ in particular, all have the correct short distance behavior to be genuine Hadamard states on $\mathbb{M}$ ([19]). However, it is easy to see ([20]) that ${\mu_L}^M$ fails to satisfy the positivity condition Eq. (22) on $\mathbb{M}$, hence it fails to be a quantum state to begin with. The reason is simple: there exist many nonzero $f \in C_0^\infty(\mathbb{M})$ such that $E_M f \neq 0$ and $\mathcal{W}(f) = 0$; in other words, there exist nonzero $F \in S(\mathbb{M})$ such that $F^W = 0$. Thus, for each such $F \in S(\mathbb{M})$, $F \neq 0$ and ${\mu_L}^M[F,F] = 0$, and this contradicts the positivity inequality on $\mathbb{M}$.

The next natural question to ask is whether ${w_L}^M$ is the regularized two-point function of a Hadamard state on $\mathbb{M}$. In other words, given a smooth, symmetric bi-solution w_L on $\mathbb{L}$ which satisfies the inequality Eq. (26), is the two-point function $\mu_{0M} + {w_L}^M$ a Hadamard state on Minkowski spacetime? This two-point function has the Hadamard form by definition, and, physically, would make sense as the "unwrapping" of the quantum state $\mu_{0L} + w_L$ onto the covering space $\mathbb{M}$. Clearly, the argument above proving ${\mu_{0L}}^M + {w_L}^M$ to be in violation of positivity does not apply to $\mu_{0M} + {w_L}^M$. Furthermore, if $\mu_{0M} + {w_L}^M$ were indeed a state on $\mathbb{M}$, the proof of the difference inequality Eq. (15) on $\mathbb{L}$ would follow immediately from the proof of ANEC on $\mathbb{M}$: the regularized stress-energy of this "state" on Minkowski spacetime is precisely the difference

$\langle T_{ab}\rangle_L - D_{ab}$ which appears in the integrand in Eq. (15). Unfortunately (for my purposes), the two-point function $\mu_{0M} + w_L{}^M$ in general does violate the positivity condition on Minkowski space, and therefore does not represent a Minkowski quantum state for every choice of w_L. The demonstration of this is only slightly more involved than the argument I gave in the preceding paragraph (which proved that $\mu_{0L}{}^M + w_L{}^M$ violates positivity). Namely, consider the following expressions for the two-point distributions μ_{0M} and $\mu_{0L}{}^M$:

$$\mu_{0M}(f,f) = \int_{-\infty}^{\infty} \frac{\pi}{|k|} dk\, |\hat{f}(k, -|k|)|^2 , \tag{27}$$

$$\mu_{0L}{}^M(f,f) = \sum_{n=-\infty}^{\infty}{}' \frac{\pi}{|k_n|} (\Delta k)\, |\hat{f}(k_n, -|k_n|)|^2 , \qquad f \in C_0^\infty(\mathbb{M}) , \tag{28}$$

where $f(k,\omega)$ demotes the Fourier transform

$$\hat{f}(k,\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(kx+\omega t)} f(x,t)\, dx\, dt$$

of $f \in L'_e{}^m(M)$, $k_n = 2\pi n/L$, and $\Delta k \equiv k_{n+1} - k_n = 2\pi/L$. The prime on the summation sign in Eq. (28) indicates that the sum excludes $n = 0$; this is because massless quantum field theory in two dimensions is formulated with test functions $f \in C_0^\infty(\mathcal{M})$ which satisfy $\int f = 0$ to avoid infrared-divergence problems. [See the paragraph following Eq. (17) in [4] for an explanation of this. Note that I ignore this complication almost completely throughout my analysis in this paper because accommodating it would not make any difference (except for making my notation more complicated than it already is) in the flow of my argument.] Now the bi-solution w_L is required to satisfy only the inequality Eq. (26), so that

$$(\mu_{0L}{}^M + w_L{}^M)(f,f) \geqslant 0 \qquad \forall f \in C_0^\infty(\mathbb{M}) , \tag{29}$$

whereas to be the regularized two-point function of a quantum state on Minkowski space it would need to satisfy

$$(\mu_{0M} + w_L{}^M)(f,f) \geqslant 0 \qquad \forall f \in C_0^\infty(\mathbb{M}) . \tag{30}$$

It is therefore sufficient to show that there exist w_L which satisfy Eq. (29) but violate Eq. (30). [Note that there exist many w_L whit.11 satisfy both Eqs. (29) and (30); any w_L which, as a bi-solution, decomposes into the tensor product of a solution with itself is an example of this. In fact, for such a two-particle state on $\mathbb{L}$, the "unwrapping" $\mu_{0M} + w_L{}^M$ *does* correspond to a genuine state on Minkowski spacetime.] To see that this is indeed the case, consider those $f \in C_0^\infty(\mathbb{M})$ whose Fourier transforms $\hat{f}(k, \omega)$ are sharply peaked around the quantized frequencies $k = k_n$, $\omega = -|k_n|$. [To find a compact supported f of this kind, start with a $\hat{f}(k,\omega)$ in $L^2(\mathbb{R}^2)$ which is such that (i) for fixed ω_0, $\hat{f}(k, \omega_0)$ decays at infinity faster than any inverse polynomial and is the restriction to $\mathbb{R}$ of an entire function $\hat{f}_1(z)$ on $\mathbb{C}$, and, similarly, for fixed k_o, $\hat{f}(k_0,\omega)$ decays at infinity faster than any inverse polynomial and is the restriction to $\mathbb{R}$ of an entire function $\hat{f}_2(z)$ on $\mathbb{C}$, (ii) at complex infinity, $\hat{f}_1(z)$ has the asymptotic behavior $|f_1(z)| \leqslant M_1 e^{a_1|z|}$, and, similarly, $f_2(z)$ has the asymptotic behavior $|\hat{f}_2(z)| \leqslant M_2 e^{a_2|z|}$, $a_i > 0$, $M_i > 0$, and (iii) $\hat{f}(k,\omega)$ is sharply peaked around $(k_n, -|k_n|)$, $n \in \mathbb{Z}$. The inverse Fourier transform $f(x,t)$ is then guaranteed to be C^∞ and of compact support by Theorem 7.4 in Chapter 6 of [21].] According to Eqs. (27)- (28), when smeared with these f, $\mu_{0L}{}^M(f,f)$ becomes arbitrarily large in comparison with $\mu_{0M}(f,f)$ as $f(k,\omega)$ get more and more sharply peaked around $(k_n, -|k_n|)$. It is then clear that a w_L can be found such that $\mu_{0L}{}^M + w_L{}^M$ satisfies Eq. (29) for all $f \in C_0^\infty(\mathbb{M})$, but for these "spiky" f peaked at $(k_n, -|k_n|)$, $w_L{}^M(f,f)$ manages to be more negative than $-\mu_{0M}(f,f)$ and hence violates Eq. (30).

Although the obvious "easy" proof of the difference inequality Eq. (15) seems to be ruled out by the above argument, the results of this section already provide all the necessary extra ingredients which, when combined with the proof of ANEC in Sect.4 of [4], constitute a complete proof of Eq. (15) as I will now explain.

4. Proof of the difference inequality in flat $S^1 \times \mathbb{R}$ spacetime

As this section follows Sect.4 of [4] very closely, for brevity I will use the prefix 4 to denote equations in [4]; e.g., Eq. (4.30) will denote Eq. (30) of Reference [4]. Let $\mu_{0L} + w_L$ be a Hadamard state on $\mathbb{L}$, of the form Eq. (23), and let γ be a complete null geodesic in $\mathbb{L}$. Just as the two-point distributions on $\mathbb{L}$ were pulled back to $\mathbb{M}$ in the previous section, so can the null geodesic γ be lifted to a complete null geodesic $\tilde{\gamma}$ on M, I can then carry out my analysis entirely on Minkowski spacetime as in Sect. 4 of [4]. Assume, without loss of generality, that $\tilde{\gamma} = \{u = 0\}$. Define

$$Y_L(v,v') \equiv \frac{\partial^2}{\partial v \partial v'} w_L{}^M(0,v,0,v') \ . \tag{31}$$

It is then obvious that along $\tilde{\gamma}$,

$$(\langle T_{ab}(v)\rangle_L - D_{ab})k^a k^b = (\langle T_{vv}\rangle_L - D_{vv}) = Y_L(v,v) \ , \tag{32}$$

where $\langle T_{ab}(v)\rangle_L$ denotes the regularized stress-energy along γ pulled back to $\tilde{\gamma}$, and D_{ab} is the stress-energy tensor [Eq. (14)] of the Casimir vacuum state μ_{0L}, again pulled back to $\tilde{\gamma}$. In precise but cumbersome notation (which therefore I will avoid) these quantities really should be written as $\pi^*(\langle T_{ab}\rangle_L)$ and $\pi^* D_{ab}$, where $\pi : \mathbb{M} \to \mathbb{L}$ is the canonical projection. Now, by applying exactly the same arguments as those in [4] leading to Eq. (4.30), I deduce from the positivity inequality Eq. (26) that for all functions $F_1, F_2 \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} &\left[2\mu_{0L}{}^M[F_1,F_1] + 8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y_L(v,v')F_1(v)F_1(v')\,dv\,dv'\right] \times \\ &\times \left[2\mu_{0L}{}^M[F_2,F_2] + 8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y_L(v,v')F_2(v)F_2(v')\,dv\,dv'\right] \\ &\geqslant |\sigma_L{}^M[F_1,F_2]|^2 \ , \end{aligned} \tag{33}$$

where now instead of Eqs. (4. 26) and (4.31) have, in accordance with Eq. (28),

$$\mu_{0L}{}^{M}[F, F] = 2\sum_{n=0}^{\infty} k_n \, |\hat{F}(k_n)|^2 \, \Delta k \,, \tag{34}$$

and

$$\sigma_L{}^{M}[F_1, F] = -4 \, \mathrm{Im} \sum_{n=0}^{\infty} k_n \overline{\hat{F}_1(k_n)} \hat{F}_2(k_n) \, \Delta k \tag{35}$$

I will now trace the arguments in [4] following Eq. (4. 31) and verify that they lead to a proof of the difference inequality 1 Eq. (15) as promised. First assume, as in [4], that the function $Y_L(v, v')$ belongs to Schwartz spat.c, i.e., it and all its derivatives decay at infinity faster than any polynomial. By the same algebra that leads in [4] to Eq. (4. 32), it follows from Eqs. (33)- (35) that for all $F_1, F_2 \in C_0^{\infty}(\mathbb{R})$

$$\left[\sum_{n=0}^{\infty} k_n |\hat{F}_1(k_n)|^2 \, \Delta k + \xi(\hat{F}_1, \hat{F}_1) - \eta(\hat{F}_1, \hat{F}_1)\right] \times$$
$$\times \left[\sum_{n=0}^{\infty} k_n |\hat{F}_2(k_n)|^2 \, \Delta k + \xi(\hat{F}_2, \hat{F}_2) - \eta(\hat{F}_2, \hat{F}_2)\right]$$
$$\geqslant \left[\mathrm{Im} \sum_{n=0}^{\infty} k_n \overline{\hat{F}_1(k_n)} \hat{F}_2(k_n) \, \Delta k\right]^2 \,, \tag{36}$$

where $\xi(\hat{F}, \hat{F})$ and $\eta(\hat{F}, \hat{F})$ are given by the same expressions as Eqs. (4. 33) (with $\hat{Y}$ replaced with $\hat{Y}_L$). Then, precisely the same argument which in [4] leads from Eq. (4. 32) to Eqs. (4. 39) and (4. 40) leads here to the cone.lusio]ls

$$\hat{Y}_L(\kappa, -\kappa) \geqslant 0 \qquad \forall \kappa \in [0, \infty) \,, \tag{37}$$

and, as in Eq. **(4.40),**

$$\int_{-\infty}^{\infty} Y_L(v, v) \, dv = 2 \int_0^{\infty} \hat{Y}_L(k, -k) \, dk \,, \tag{38}$$

which, when combined with Eq. (32), prove the difference inequality Eq. (15) in this Schwartz-s pace $Y_L(v, v')$ case [in which the integrand $(\langle T_{ab}\rangle_L - D_{ab})k^a k^b$ is integrable]. In the general case, the argument I need to use is again identical to the one in [4] between Eqs. (4, 40) and (4. 59), except that in the present analysis it leads to equalities of the form

$$\begin{aligned} \mu_{0L}{}^{M}[F_{\lambda,\kappa}, F_{\lambda,\kappa}] &= \beta(\lambda, \kappa)\left[\tfrac{1}{2}\lambda\kappa + \epsilon_1(\lambda\kappa)\right] \\ \mu_{0L}{}^{M}[G_{\lambda,\kappa}, G_{\lambda,\kappa}] &= \beta(\lambda, \kappa)\left[\tfrac{1}{2}\lambda\kappa + \epsilon_2(\lambda\kappa)\right] \\ \sigma_L{}^{M}[F_{\lambda,\kappa}, G_{\lambda,\kappa}] &= \beta(\lambda, \kappa)\left[\lambda\kappa + \epsilon_3(\lambda\kappa)\right] \end{aligned} \tag{39}$$

With the same choices for $F_{\lambda,\kappa}(v)$ and $G_{\lambda,\kappa}(v)$ as in Eqs. (4.42). Here $\beta(\kappa, \lambda)$ is a continuous function such that

$$0 < \beta(\lambda, \kappa) < 1 \qquad \forall \, \lambda > 0, \, \kappa \geqslant 0 \,, \tag{40}$$

and $\epsilon_l(x)$ are continuous functions with the same decay property as described in [4] following Eq. (4. 46). The second part of the argument [spelled out in [4] between Eqs. (4.48) and (4. 59)] can be repeated identically here, leading to the inequality

$$\int_{-\infty}^{\infty} (\langle T_{vv}\rangle_L - D_{vv})[c(v/\lambda)]^2\, dv \geqslant \frac{1}{\lambda}\int_{0}^{\infty} \beta(\lambda, y/\lambda)\, \epsilon_{10}(y)\, dy\,. \qquad (41)$$

By Eq. (40) and the asymptotic behavior of $\epsilon_l(x)$, there exist $\delta, K > 0$ such that

$$\int_{0}^{\infty} \beta(\lambda, y/\lambda)\, \epsilon_{10}(y)\, dy \geqslant -\int_{0}^{\infty} \frac{K}{(1+y)^{1+\delta}}\, dy \equiv -C\,, \qquad C > 0\,, \qquad (42)$$

and, when combined with Eq. (41), this inequality proves not only the difference-inequality result

$$\liminf_{\lambda\to\infty} \int_{-\infty}^{\infty} (\langle T_{vv}\rangle_L - D_{vv})[c(v/\lambda)]^2\, dv \geqslant 0\,, \qquad (43)$$

but also the sharper estimate

$$\int_{-\infty}^{w} (\langle T_{vv}\rangle_L - D_{vv})[c(v/\lambda)]^2\, dv \geqslant -\frac{C}{\lambda} \qquad (44)$$

as in Eq. *(4.59)*,

5. Proof of the difference inequality in curved two-dimensional spacetime

Consider a two-dimensional, globally hyperbolic spacetime $(\mathcal{M}, g)$ with a massless Klein-Gordon field ϕ, and let γ be a complete null geodesic in $\mathcal{M}$. By Sect. 5 of [4], if γ is achronal the renormalized stress-cllcrgy tensor in every Hadamard state of ϕ satisfies the difference inequality Eq. (15) along γ with $D_{ab} = O$ *(i.e.,* it satisfies ANEC). If γ is not achronal, then 1 claim that $(\mathcal{M}, g)$ has topology S' X $\mathbb{R}$ and is globally conformal to a flat cylinder $\mathbb{I}_L$ for some $L > 0$.

To prove this, let p and g be any pair of timelike-related points along γ (such pairs exist since γ is assumed non-achronal). Assume that $q \in I^+(p)$. Since $(\mathcal{M}$, g) is globally hyperbolic, it is causally simple ([22]), i.e., $\dot{J}^+(p)$ consists of null **geodesics** from p which have no past endpoints other than p itself. Clearly, q cannot belong to $\dot{J}^+(p)$ as it is timelike-related to p; therefore, γ must leave $\dot{J}^+(p)$ at some point p' to the past of q. Since $\dot{J}^+(p)$ is an achronal C^{1-} submanifold without boundary ([22], Chapter 6), and p' lies on $\dot{J}^+$ (p), γ must intersect the Other null generator δ of $\dot{J}^+(p)$ at p'; otherwise p' would lie on the boundary of $\dot{J}^{(+)}(p)$. The portion of δ to the future of p' is timelike related to p (since every point in this portion lies on a broken null geodesic from p), therefore δ must also leave $\dot{J}^+(p)$ at p'; otherwise $\dot{J}^+(p)$ would not be achronal. Consequently, $\dot{J}^+(p)$ is compact. Because $(\mathcal{M}$, g) is globally hyperbolic, there exists a Cauchy surface Σ through p, and any global timelike vector field on $\mathcal{M}$ provides a diffeomorphism from $\dot{J}^+(p)$ into Σ. Since $\dot{J}^+(p)$ is compact without boundary, Σ must also be compact. But the only compact 1-manifold is

S^1, and by global hyperbolicity $\mathcal{M}$ is diffeomorphic to $\Sigma \times \mathbb{R}$, therefore, $\mathcal{M}$ is diffeomorphic to $S^1 \times \mathbb{R}$. To prove that $(\mathcal{M}, g)$ is globally conformal to $\mathbb{L}$, it suffices to simply carry out the usual local argument which proves that any two-dimensional spacetime is locally conformally flat, using as the time coordinate t a smooth labeling of the Cauchy surfaces $S^1 \subset \mathcal{M}$, and as the x coordinate any coordinate on one of the Cauchy S^1's extended globally onto $\mathcal{M}$ (apart from the obvious coordinate singularity on S^1) by keeping it constant along a timelike vector field orthogonal to the Cauchy surfaces. To summarize: if a globally hyperbolic $(\mathcal{M}, g)$ admits a non-achronal null geodesic, then there exists a diffeomorphism $\Psi : \mathbb{L} \to \mathcal{M}$ such that $\Psi^* g = C(u, v)\, \eta_L$, where $\eta_L = -du\, dv$ is the flat metric on $\mathbb{L}$ written in local null coordinates $\{u, v\}$, and $C(u, v) > 0$ is a smooth function on $\mathbb{L}$.

With γ a complete non-achronal null geodesic in $(\mathcal{M}, g)$, and with the simple geometry of $\mathcal{M}$ as uncovered in the above paragraph, it is now quite straightforward to give an analysis parallel to that of Sect. 5 in [4], where the proof of ANEC in curved two-dimensional spacetime was reduced to the preceding proof in flat Minkowski space. Namely, the Casimir vacuum state on $\mathbb{L}$ given by Eq. (28) determines, under the conformal isometry $\Psi : \mathbb{L} \to \mathcal{M}$, a corresponding quantum state on $\mathcal{M}$. [This is because the massless wave operator as well as the Klein-Gordon inner product are conformally invariant in two dimensions, consequently there exists a one-to-one correspondence between states defined on $\mathbb{L}$ and states defined on $\mathcal{M}$, determined by mapping the two-point distributions backwards or forwards via the diffeomorphism Ψ. Note also that the spacetime $(\mathcal{M}, g)$ is in fact *isometric* to the cylinder $\mathbb{L}$ equipped with the metric $g = C\, \eta_L = -C(u, v)\, du\, dv$, therefore I can use this representation of $(\mathcal{M}, g)$ throughout without any loss of generality, and as this will simplify my notation considerably I will do so from here on.] Let me denote the two-point distribution of this state by $\mu_c(x, x')$ [in the isometric representation of $\mathcal{M}$ as $(\mathbb{L}, g)$, this distribution has the same functional form as $\mu_{0L}(x, x')$]. The renormalized stress-energy in this "conformal" Casimir vacuum is determined entirely by the conformal anomaly [see Eqs. (10)-(11) in [3]], and can be written in the form

$$D_{ab} = \frac{1}{C} D^{(0)}_{ab} + \frac{1}{48\pi} R g_{ab} + \theta_{ab}\,, \tag{45}$$

where $D^{(0)}_{ab}$ is the Casimir energy on the flat $\mathbb{L}$ given by the right hand side of Eq. (14), R is the Ricci scalar of $(\mathcal{M}, g)$, and, in the local null coordinates $\{u, v\}$,

$$\begin{aligned}
\theta_{uu} &= \frac{1}{24\pi}\left[\frac{C_{,uu}}{C} - \frac{3}{2}\frac{{C_{,u}}^2}{C^2}\right], \\
\theta_{vv} &= \frac{1}{24\pi}\left[\frac{C_{,vv}}{C} - \frac{3}{2}\frac{{C_{,v}}^2}{C^2}\right], \\
\theta_{uv} &= \theta_{vu} = 0\,.
\end{aligned} \tag{46}$$

Now, any Hadamard state on $(\mathcal{M}, g)$ has a two-point function of the form

$$\mu(x, x') = \mu_c(x, x') + w(x, x')\,, \tag{47}$$

where $w(x, x')$ is a smooth bi-solution such that μ satisfies the positivity inequality Eq. (19). It is crucial to keep in mind that although $\mu_c(x, x')$ has the

Hadamard form, it is *not* a locally constructed two-point distribution, hence it cannot be used to regularize $\mu(x, x')$ as Eq. (47) suggests. Instead, an appropriate Hadamard distribution) μ_H constructed entirely out of the local geometry of $(\mathcal{M}, g)$ needs to be subtracted from μ to obtain the regularized two-point function; the components of the stress-energy tensor are then obtained as the coincidence limits of the derivatives of this regularized two-point function $\mu - \mu_H$. [This point is of course also valid for the analysis of the previous section, where it was implicit in the derivation of Eq. (32) from Eq. (31).] However, for my analysis here (as also for the analysis of the previous section), it is not necessary to make explicit the form of $\mu_H(x, x')$; the stress-energy due to the difference $\mu_c - \mu_H$ is already determined completely by the Casimir contribution Eqs. (45)-(46), and the rest of the stress-energy is given simply by coincidence limits of the appropriate derivatives of the smooth bi-solution $w(x, x')$. Therefore, expressions of the form Eqs. (31)- (32) (with w replacing WI,) are still valid for the difference-inequality integrand $(\langle T_{ab}\rangle - D_{ab})k^a k^b$ along γ, and using the positivity inequality for the two-point, function Eq. (47) in exactly the same manner as I did in the previous section, I arrive at the following final conclusion:

Theorem: Let $(\mathcal{M}, g)$ be a globally hyperbolic two-dimensional spacetime with a massless Klein-Gordon field ϕ. Then the regularized stress-energy tensor $\langle\omega|T_{ab}|\omega\rangle$ of ϕ in any Hadamard state ω is constrained in the following way: (i) Along every *achronal* complete null geodesic $\gamma \subset \mathcal{M}$ the difference inequality Eq. (15) holds with $D_{ab} = 0$. (ii) If $\mathcal{M}$ admits a non-achronal null geodesic, then $(\mathcal{M}, g)$ is globally conformal to II, $\cong S^1 \times \mathbb{R}$, and along every complete null geodesic γ in $\mathcal{M}$ the difference inequality Eq. (15) holds with D_{ab} defined by Eqs. (45)- (46).

The proof of the difference inequality in two dimensions suggests that more generally, when $(\mathcal{M}, g)$ is a multiply-connected (four-dimensional) globally hyperbolic spacetime, an inequality of the form Eq. (15) might hold on a complete non-achronal null geodesic γ if the lifting of γ in the simply-connected covering space $\mathcal{M}$ is achronal and satisfies ANEC. When γ is a non-achronal complete null geodesic in a *simply-connected* spacetime (i. e., when the failure of achronality is due to gravitational focusing rather than the topology of $\mathcal{M}$), my proof does not provide any insights into whether difference inequalities are reasonable as constraints on the stress-energy tensor along γ.

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the extrinsic. curvature is calculated with respect to the outward-pointing normal, and consequently has a positive sign when the boundary is convex, and a negative sign when it is concave. If the conducting surface contains a flat direction, the divergent contribution projected on that direction vanishes (due to conformal invariance). It is then clear that along any static, complete timelike geodesic that lies at rest sufficiently c.lose to a concave conductor the averaged weak energy condition will be violated. Because of the geometry, no complete *null* geodesic can ever come arbitrarily close to a concave conductor. A complete null geodesic can graze arbitrarily closely by a *convex* boundary, or a boundary which contains a flat direction (and then only along that direction); but in these cases the divergent contribution from the stress-cllergy is nonnegative. In fact, ANEC holds; in the static, conformal vacuum state, in which the regularized two-point function $w(x,x')$ satisfies appropriate asymptotic fall-off conditions, the analysis given in [4] modified for conformal coupling- is sufficient to prove ANEC along complete null geodesics in Minkowski spacetime in the presence of (stationary) conducting boundaries. The idea of using conducting boundaries in flat spacetime as a testing ground for averaged energy conditions is originally duc to Curt Cutler.

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